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Concept 1 | Binomial and Multinomial Theorems

LESSON NOTES 1 • BINOMIAL THEOREM

Learning Objectives

By the end of this lesson, you will be able to:

- calculate binomial coefficients using ${}_nC_r$.
- expand $(a+b)^n$ using the Binomial Theorem.
- find a specified term or coefficient without full expansion.
- recognize the pattern of powers and coefficients.

Why do we need the theorem?

Repeated multiplication becomes lengthy when the exponent is large. The Binomial Theorem gives every term of $(a+b)^n$ directly, using combinations and a predictable pattern of powers.

Key idea: *every term has total degree n.*

The Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n} {}_nC_r\, a^{n-r}\, b^r$$

General term: $T_{r+1} = {}_nC_r\, a^{n-r}\, b^r$

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Pattern Check

- There are $n+1$ terms.
- The power of a decreases from n to 0.
- The power of b increases from 0 to n.
- The two exponents in each term add to n.
- ${}_nC_r = {}_nC_{n-r}$.

1 Example 1 | Calculate a binomial coefficient

Evaluate ${}^7C_2$.

$${}^7C_2 = \frac{7!}{2!\,5!} = \frac{7\times 6}{2\times 1} = 21$$

3 Example 3 | Find a specified coefficient

Find the coefficient of x^3 in $(x+2)^5$.

For x^3, $n-r=3$, so $r=2$.

Coefficient $= {}^5C_2(2)^2 = 10\times 4 = 40$

2 Example 2 | Expand a binomial

Expand $(2x-3)^4$.

Use coefficients 1, 4, 6, 4, 1.

$$(2x-3)^4 = (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$$
$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Watch Out: keep the *negative sign inside the powered term.*

Try It Yourself

Answer without using repeated multiplication.

a. Evaluate ${}^8C_3$.

b. Expand $(x-2)^4$.

c. Find the coefficient of x^2 in $(2x+1)^5$.

d. Find the 5^{th} term of $(a+3b)^6$.

Concept 1 | Binomial and Multinomial Theorems

LESSON NOTES 2 • SPECIAL TERMS AND COEFFICIENTS

Term Number and the Value of r

In T_{r+1}, the term number is $r+1$.

kth term $\Rightarrow$ $r = k - 1$

4th term $\Rightarrow r = 3$ 7th term $\Rightarrow r = 6$

A Reliable Method

1. Write the general term.
2. Match the term number or the required power.
3. Solve for r, then check $0 \le r \le n$.
4. Substitute and simplify carefully.

4 Example 4 | Find a specified term

Find the 4^{th} term of $(2x - y)^6$.

Since $r = 4 - 1 = 3$,

$$T_4 = {}^6C_3(2x)^3(-y)^3$$
$$= 20 \cdot 8x^3 \cdot (-y^3)$$
$$= \mathbf{-160x^3y^3}$$

5 Example 5 | Find a constant term

Find the constant term in $\left(x^2 + \frac{2}{x}\right)^6$.

$$T_{r+1} = {}^6C_r(x^2)^{6-r}\left(\frac{2}{x}\right)^r$$
$$= {}^6C_r\, 2^r x^{12-3r}$$

For a constant term: $12 - 3r = 0$, so $r = 4$.

Constant term $= {}^6C_4 \cdot 2^4 = 15 \cdot 16 = \mathbf{240}$

6 Example 6 | Find the middle term

Find the middle term of $(a + 2b)^8$.

There are $8 + 1 = 9$ terms, so the middle term is the 5^{th}.

For T_5, $r = 4$.

$$T_5 = {}^8C_4\, a^4(2b)^4 = 70 \cdot 16a^4b^4$$
$$= \mathbf{1120a^4b^4}$$

7 Example 7 | Find the sum of coefficients

Find the sum of the coefficients of $(3x - 2)^5$.

Let $P(x) = (3x - 2)^5$. Put $x = 1$.

$$P(1) = (3 - 2)^5 = 1$$

Therefore, the sum of the coefficients is **1**.

Watch Out: the kth term uses $r = k - 1$, not $r = k$.

Try It Yourself

Show the essential steps.

a. Find the 5^{th} term of $(x - 2y)^7$.

b. Find the constant term in $\left(x^3 + \frac{1}{x}\right)^8$.

c. Find the middle term of $(2m + n)^6$.

d. Find the sum of the coefficients of $(4x - 3)^6$.

Concept 1 | Binomial and Multinomial Theorems

LESSON NOTES 3 • MULTINOMIAL THEOREM

The Multinomial Theorem

For non-negative integers p, q and r whose sum is n:

$$(x + y + z)^n = \sum_{p+q+r=n} \frac{n!}{p!q!r!} x^p y^q z^r$$

where $p + q + r = n$.

Multinomial coefficient $= \dfrac{n!}{p!q!r!}$

How to Find a Required Term

1. Match p, q and r with the required powers.
2. Check that $p + q + r = n$.
3. Calculate $\dfrac{n!}{p!q!r!}$.
4. Include numerical factors and their signs.

Number of distinct terms in a trinomial expansion $= C(n + 2, 2)$

8 Example 8 | Expand a simple trinomial

Expand $(x + y + z)^2$.

$$(x + y + z)^2$$
$$= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

Every term has total degree 2.

9 Example 9 | Find a specified coefficient

Find the coefficient of x^2yz^3 in $(x + y + z)^6$.

Here $p = 2$, $q = 1$ and $r = 3$; $2 + 1 + 3 = 6$.

$$\text{Coefficient} = \frac{6!}{2!1!3!}$$
$$= \frac{720}{2 \cdot 1 \cdot 6} = 60$$

10 Example 10 | Include numerical factors

Find the term containing a^2bc in $(2a - b + 3c)^4$.

Use $p = 2$, $q = 1$ and $r = 1$.

$$\text{Term} = \left[\frac{4!}{2!1!1!}\right](2a)^2(-b)(3c)$$
$$= 12 \cdot 4 \cdot (-1) \cdot 3\, a^2bc$$
$$= -144\, a^2bc$$

11 Example 11 | Count the terms

How many distinct terms are in $(x + y + z)^5$?

$$\text{Number of terms} = C(5 + 2, 2)$$
$$= C(7, 2) = 21$$

Watch Out: the required exponents must add to n; a missing variable has exponent 0.

Try It Yourself

Show the essential steps.

a. Expand $(a + b - c)^2$.

b. Find the coefficient of x^3y^2z in $(x + y + z)^6$.

c. Find the term containing ab^2c in $(2a + b - c)^4$.

d. Find the number of distinct terms in $(m + n + p)^6$.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 1 • BINOMIAL BASICS

Solve the following problems. Show clear algebraic steps when needed.

📖 = From the textbook source

1 **Evaluate each binomial coefficient.**

(a) ${}^{8}C_{2}$ (b) ${}^{9}C_{4}$ (c) ${}^{11}C_{3}$ (d) ${}^{12}C_{5}$

(e) ${}^{10}C_{7}$ (f) ${}^{13}C_{2}$ (g) ${}^{14}C_{0}$ (h) ${}^{15}C_{14}$

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2 📖 **Find the coefficient of the specified term in [ ] for each expansion.**

(a) $(2x - y)^8 \quad [x^4y^4]$ (b) $(3a + b)^4 \quad [ab^3]$

(c) $(x - 4y)^5 \quad [x^3y^2]$ (d) $(3x + 1)^5 \quad [x^4]$

(e) $(2a - 3)^4 \quad [a]$ (f) $(3x + 2)^5 \quad [x^3]$

(g) $(2x^2 - 1)^6 \quad [x^6]$ (h) $(3x^2 - 1)^5 \quad [x^4]$

(i) $(3x^2 + 2)^6 \quad [x^6]$

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3 📖 **Expand each binomial in descending powers.**

(a) $(x + 1)^4$ (b) $(a + b)^4$

(c) $(a - 2b)^4$ (d) $(x - 2)^5$

(e) $(3x + 2y)^5$ (f) $(2a^2 - 1)^5$

(g) $(x - 3)^6$ (h) $(2x - 3y)^6$

(i) $\left(x + \frac{1}{3}\right)^6$

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4 **Find the requested term, then simplify.**

(a) The 3rd term of $(x - 2)^8$ (b) The 6th term of $(2a + b)^7$

(c) The middle term of $(p - 2q)^6$ (d) The constant term in $\left(x^3 + \frac{1}{x^2}\right)^{10}$

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5 **Answer each short question.**

(a) Find the number of terms in the expansion of $(x + y)^9$. (b) Find the position of the middle term in $(a + b)^8$.

(c) Find the sum of coefficients in the expansion of $(x + 2)^5$. (d) Find the coefficient of x^0 in the expansion of $(3 + x)^4$.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 2 • COEFFICIENTS AND TERMS

Solve the following problems. Show clear algebraic steps when needed.

📖 = From the textbook source

6 📖 **Find the coefficient of the specified term in [ ] for each expansion.**

(a) $(2x - 3y)^7 \quad [x^6y]$

(b) $(3x - 2)^5 \quad [x^2]$

(c) $(3a^2 - 2)^8 \quad [a^6]$

(d) $(x + 2y)^9 \quad [x^5y^4]$

(e) $(2m - n)^8 \quad [m^3n^5]$

(f) $(1 - 3x^2)^7 \quad [x^8]$

(g) $(2x^3 + 1)^6 \quad [x^9]$

(h) $(a^2 - 2b)^5 \quad [a^6b^2]$

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7 📖 **Expand each expression in descending powers.**

(a) $(x + 2)^6$

(b) $(3a - 2b)^5$

(c) $(2x - 1)^6$

(d) $(a + 3b)^5$

(e) $(2p^2 + q)^4$

(f) $\left(x - \frac{1}{2}\right)^6$

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8 **Find the requested term.**

(a) The 5th term of $(2x - y)^9$

(b) The 4th term of $(a^2 + 3b)^7$

(c) The middle term of $(x - 2)^{10}$

(d) The two middle terms of $(p + q)^9$

(e) The term containing x^8 in $\left(x^2 + \frac{2}{x}\right)^7$

(f) The term containing y^5 in $(2x - y)^8$

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9 **Find the term independent of x in each expansion.**

(a) $\left(x + \frac{1}{x}\right)^8$

(b) $\left(x^2 - \frac{3}{x}\right)^9$

(c) $\left(2x^3 + \frac{1}{x^2}\right)^{10}$

(d) $\left(x^4 - \frac{2}{x}\right)^{10}$

(e) $\left(x^2 + \frac{2}{x^3}\right)^{10}$

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10 **Answer each question.**

(a) The coefficient of x in $(1 + kx)^6$ is 24. Find k.

(b) If $m > 0$ and the coefficient of x^2 in $(1 + mx)^5$ is 90, find m.

(c) Find the sum of all coefficients in $(2x + 3)^6$.

(d) Find the sum of all coefficients in $(3x - 2)^7$.

(e) If the coefficient of x^3 in $(1 + x)^n$ is 56, find n.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 3 • ADVANCED BINOMIALS

Solve the following problems. Show clear algebraic steps when needed.

📖 = From the textbook source

11 📖 **Use coefficient sums to answer.**

(a) If $(1+x)^5 = a + bx + cx^2 + dx^3 + ex^4 + fx^5$, find $a+b+c+d+e+f$.

(b) Find the sum of all coefficients in $(2x-1)^8$.

(c) Find the alternating sum of coefficients in $(3x+2)^6$.

(d) Find the sum of coefficients of the even powers of x in $(1+x)^{10}$.

(e) Find the sum of coefficients of the odd powers of x in $(1+x)^{10}$.

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12 **Find the unknown value.**

(a) The coefficient of x in $(1+ax)^7$ is 35. Find a.

(b) If $b>0$ and the coefficient of x^2 in $(1+bx)^6$ is 240, find b.

(c) The first three coefficients of $(1+kx)^8$ are 1, 24 and m. Find k and m.

(d) The coefficient of x^3 in $(1+x)^n$ is 120. Find n.

(e) The coefficients of x^2 and x^3 in $(1+x)^n$ are equal. Find n.

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13 **Find the specified coefficient.**

(a) $[x^4]$ in $(1+x)^5(1-x)^3$ **(b)** $[x^5]$ in $(1+2x)^4(1-x)^5$ **(c)** $[x^6]$ in $(1+x^2)^5(1-x)^4$

(d) $[x^4]$ in $(1+x+x^2)(1+x)^6$ **(e)** $[x^3]$ in $(2-x)^4(1+x)^3$

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14 **Work with the general term.**

(a) Write the general term in the expansion of $(2x-3)^9$.

(b) Find r if T_{r+1} in $(x+2)^{10}$ contains x^6.

(c) Find r if T_{r+1} in $\left(x^2+\frac{1}{x}\right)^{12}$ is independent of x.

(d) Find $\frac{T_5}{T_4}$ in the expansion of $(1+2x)^8$.

(e) Find the greatest numerical coefficient in $(1+x)^{10}$.

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15 **Begin multinomial practice.**

(a) Expand $(x+y+z)^2$.

(b) Expand $(a+b+c)^3$.

(c) Find the coefficient of xyz in $(x+y+z)^3$.

(d) Find the coefficient of x^2yz in $(x+y+z)^4$.

(e) Find the coefficient of a^2b^3c in $(a+b+c)^6$.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 4 • MULTINOMIAL BASICS

Solve the following problems. Show clear algebraic steps when needed.

16 **Evaluate each multinomial coefficient.**

(a) $\dfrac{5!}{2!\,2!\,1!}$ (b) $\dfrac{6!}{3!\,2!\,1!}$ (c) $\dfrac{7!}{4!\,2!\,1!}$

(d) $\dfrac{8!}{3!\,3!\,2!}$ (e) $\dfrac{9!}{4!\,3!\,2!}$ (f) $\dfrac{10!}{5!\,3!\,2!}$

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17 **Expand each expression.**

(a) $(x+y+z)^2$ (b) $(a+b+c)^3$ (c) $(2x+y+z)^3$

(d) $(x-y+2z)^3$ (e) $(a+2b-c)^3$ (f) $(p+q+1)^4$

---

18 **Find the coefficient of the specified term.**

(a) xyz in $(x+y+z)^3$ (b) x^2yz in $(x+y+z)^4$

(c) a^3b^2c in $(a+b+c)^6$ (d) $x^2y^3z^2$ in $(x+y+z)^7$

(e) p^4qr^3 in $(p+q+r)^8$ (f) x^3y^2 in $(2x-y+z)^5$

(g) a^2bc^2 in $(a-2b+3c)^5$ (h) xy^2z^3 in $(2x+y-z)^6$

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19 **Answer each question about the expansion.**

(a) Find the number of terms in $(x+y+z)^5$. (b) Find the number of terms in $(a+b+c+d)^4$.

(c) State the degree of every term in $(2x-y+z)^7$. (d) Find the coefficient of y^4z^2 in $(x+y+z)^6$.

(e) Does $x^4y^2z^3$ appear in $(x+y+z)^8$? Give a reason.

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20 **Find the specified coefficient or term.**

(a) $[x^4]$ in $(1+x+x^2)^4$ (b) $[x^5]$ in $(1+x+x^2)^5$

(c) $[x^6]$ in $(1+x+x^3)^6$ (d) $[x^4]$ in $(1+2x-x^2)^5$

(e) Find the constant term in $\left(x+y+\dfrac{1}{xy}\right)^6$.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 5 • MULTINOMIAL APPLICATIONS

Solve the following problems. Show clear algebraic steps when needed.

21 **Find the coefficient of the specified term.**

(a) x^3y^2z in $(2x - y + 3z)^6$

(b) $a^2b^3c^2$ in $(a + 2b - c)^7$

(c) p^4q^2r in $(3p + q - 2r)^7$

(d) $x^2y^3z^2$ in $(x - 2y + z)^7$

(e) m^3n^2 in $(2m + n + 1)^5$

(f) ab^2c^3 in $(2a - 3b + c)^6$

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22 **Work with four variables.**

(a) Find the coefficient of $abcd$ in $(a + b + c + d)^4$.

(b) Find the coefficient of x^2yzw in $(x + y + z + w)^5$.

(c) Find the coefficient of p^3q^2rs in $(p + q + r + s)^7$.

(d) Find the coefficient of $a^2b^2c^2d^2$ in $(a + b + c + d)^8$.

(e) Find the coefficient of $x^3y^2zw^2$ in $(2x - y + z + 3w)^8$.

(f) Find the coefficient of a^2bc^3d in $(a + 2b - c + d)^7$.

---

23 **Count the terms.**

(a) Find the number of terms in $(x + y + z)^8$.

(b) Find the number of terms in $(a + b + c + d)^5$.

(c) Find the number of terms in $(p + q + r + s + t)^4$.

(d) How many terms contain all three variables in $(x + y + z)^6$?

(e) How many terms are independent of z in $(x + y + z)^7$?

(f) How many terms contain x^2 in $(x + y + z)^8$?

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24 **Use substitutions to find coefficient sums.**

(a) Find the sum of all coefficients in $(x + 2y - 3z)^7$.

(b) Find the sum of all coefficients in $(2a - b + c)^8$.

(c) Find the sum of coefficients of terms with even powers of x in $(x+y+z)^6$.

(d) Find the sum of coefficients of terms with odd powers of x in $(x+y+z)^6$.

(e) Find the sum of coefficients of terms independent of z in $(2x - y + z)^5$.

(f) Find the sum of coefficients of terms containing z^2 in $(x + y + z)^6$.

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25 **Find the unknown value.**

(a) The coefficient of x^2y^2z in $(x + y + kz)^5$ is 120. Find k.

(b) If $m > 0$ and the coefficient of a^3b^2 in $(2a + mb + 1)^5$ is 320, find m.

(c) The coefficient of xyz in $(x + y + z)^n$ is 6. Find n.

(d) The expansion of $(a + b + c)^n$ has 28 terms. Find n.

(e) If $k > 0$ and the coefficient of $x^4y^2z^2$ in $(x+y+kz)^8$ is 1680, find k.

(f) The expansion of $(x + y + z + w)^n$ has 84 terms. Find n.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 6 • HIGHER-ORDER PROBLEMS

Solve the following problems. Show clear algebraic steps when needed.

26 **Find the specified coefficient.**

(a) $[x^6]$ in $(1+x+x^2)^8$ (b) $[x^7]$ in $(1+2x-x^2)^6$ (c) $[x^8]$ in $(1-x+x^3)^7$

(d) $[x^5]$ in $(2+x+x^2)^5$ (e) $[x^9]$ in $(1+x^2+x^3)^6$ (f) $[x^4]$ in $(1-2x+3x^2)^4$

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27 **Find the term independent of the variables.**

(a) $\left(x+y+\frac{1}{xy}\right)^9$ (b) $\left(x^2+y^2+\frac{1}{xy}\right)^8$ (c) $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^6$

(d) $\left(2x+\frac{1}{x}+1\right)^8$ (e) $\left(x^2-\frac{2}{x}+3\right)^6$

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28 **Work with the general multinomial term.**

(a) Write the general term in the expansion of $(x+2y-3z)^n$.

(b) Find the coefficient of $x^3y^2z^4$ in $(x+y+z)^9$.

(c) The coefficient of $x^4y^3z^2$ in $(x+ky+z)^9$ is 10080. Find k.

(d) Find the term containing equal powers of x, y and z in $(x+y+z)^{12}$.

(e) Find the coefficient of the term containing equal powers of a, b, c and d in $(a+b+c+d)^8$.

---

29 **Find the greatest coefficient.**

(a) Find the greatest coefficient in $(x+y+z)^6$.

(b) Find the greatest coefficient in $(x+y+z)^7$.

(c) Find the greatest coefficient in $(a+b+c+d)^8$.

(d) Identify every term with the greatest coefficient in $(x+y+z)^8$.

(e) Compare the coefficients of $x^4y^2z^2$ and $x^3y^3z^2$ in $(x+y+z)^8$.

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30 **Verify each statement using the theorem.**

(a) The sum of all coefficients in $(x+y+z)^3$ is 27.

(b) The expansion of $(a+b+c+d)^4$ has 35 terms.

(c) The sum of coefficients of terms with even powers of x in $(x+y+z)^5$ is 122.

(d) The coefficients of x^4y^2 and x^2y^4 in $(x+y+z)^6$ are equal.

(e) If A is the sum of coefficients of terms with even powers of x and B is the sum for odd powers in $(x+y+z)^6$, find $A-B$.

Concept 1 | Binomial and Multinomial Theorems

PRACTICE 7 • MIXED REVIEW

Solve the following problems. Show clear algebraic steps when needed.

31 Choose the correct answer.

(a) $[x^3]$ in $(1 + 2x)^5$ = [40, 80, 160, 320]

(b) The middle term of $(x + y)^8$ is $[T_4, T_5, T_6, T_7]$

(c) The number of terms in $(a + b + c)^6$ is [21, 28, 36, 45]

(d) The coefficient of x^2yz in $(x + y + z)^4$ is [6, 12, 18, 24]

(e) The sum of all coefficients in $(2x - 1)^7$ is [−1, 0, 1, 128]

(f) The greatest coefficient in $(x + y + z)^6$ is [60, 90, 120, 180]

32 Complete each statement.

(a) The coefficient of $a^{n-r}b^r$ in $(a + b)^n$ is _____.

(b) The number of terms in $(x + y)^n$ is _____.

(c) The number of terms in $(x + y + z)^n$ is _____.

(d) The number of terms in $(a + b + c + d)^n$ is _____.

(e) The sum of all coefficients of $P(x, y, z)$ is found by evaluating _____.

(f) The alternating sum with respect to x is found by evaluating _____.

33 Write True or False. Correct each false statement.

(a) $[x^4]$ in $(1 + x)^6$ is 15.

(b) The expansion of $\left(x + \frac{1}{x}\right)^5$ has a constant term.

(c) The expansion of $(a + b + c)^4$ has 15 terms.

(d) The coefficient of xyz in $(x + y + z)^3$ is 6.

(e) The greatest coefficient in $(x + y + z)^5$ is 30.

(f) The sum of all coefficients in $(x - y + z)^6$ is 1.

34 Solve each mixed problem.

(a) Find $[x^4]$ in $(1 + x)^4(1 - x)^3$.

(b) Find the constant term in $\left(x^2 + \frac{1}{x}\right)^9$.

(c) Find the coefficient of a^2b^2c in $(2a - b + c)^5$.

(d) Find the number of terms in $(a + b + c + d)^6$.

(e) Find the sum of coefficients of terms with even powers of x in $(x + y + z)^7$.

(f) Find the coefficient of x^3y^2z in $(x + y + 2z)^6$.

35 Answer the higher-order questions.

(a) If ${}^nC_2 = 66$, find n.

(b) If $k > 0$ and the coefficient of x^2y^2z in $(x + y + kz)^5$ is 180, find k.

(c) The expansion of $(a + b + c)^n$ has 45 terms. Find n.

(d) Prove that the coefficients of $x^{n-r}y^r$ and x^ry^{n-r} in $(x + y)^n$ are equal.

(e) Compare the greatest coefficients in $(x + y + z)^7$ and $(x + y + z)^8$.

Concept 2 | Polynomial Division

LESSON NOTES 1 • DIVISION BASICS

Learning Objectives

- Identify the dividend, divisor, quotient, and remainder.
- Divide a polynomial by a monomial or by another polynomial.
- Check a result using the division identity.

Key Idea

For polynomials $P(x)$ and $D(x)$, where $D(x) \neq 0$:

$$P(x) = D(x)Q(x) + R(x)$$

Dividend = Divisor × Quotient + Remainder, where deg $R(x)$ < deg $D(x)$.

Polynomial Long Division

1. Arrange terms in descending powers.
2. Insert zero terms for any missing powers.
3. Divide the leading terms.
4. Multiply, subtract, and bring down.
5. Repeat until the remainder has lower degree than the divisor.

Worked Example 1 • Dividing by a monomial

Divide $(12x^4 - 18x^3 + 6x^2)$ by $6x^2$.

$$(12x^4 - 18x^3 + 6x^2) \div 6x^2$$

$$= \frac{12x^4}{6x^2} - \frac{18x^3}{6x^2} + \frac{6x^2}{6x^2}$$

$$= 2x^2 - 3x + 1$$

Worked Example 2 • Long division with no remainder

Divide $2x^3 + 3x^2 - 8x - 12$ by $x + 2$.

$$\begin{array}{r} 2x^2 - x - 6 \\ x+2 \overline{)\ 2x^3 + 3x^2 - 8x - 12} \\ \underline{-(2x^3 + 4x^2)} \\ -x^2 - 8x \\ \underline{-(-x^2 - 2x)} \\ -6x - 12 \\ \underline{-(-6x - 12)} \\ 0 \end{array}$$

$Q(x) = 2x^2 - x - 6, \quad R(x) = 0$

Check: $(x + 2)(2x^2 - x - 6) = 2x^3 + 3x^2 - 8x - 12$

Worked Example 3 • A missing term and a remainder

Divide $x^3 - 5x + 4$ by $x - 2$.

Write $x^3 + 0x^2 - 5x + 4$ before dividing.

$$\begin{array}{r} x^2 + 2x - 1 \\ x-2 \overline{)\ x^3 + 0x^2 - 5x + 4} \\ \underline{-(x^3 - 2x^2)} \\ 2x^2 - 5x \\ \underline{-(2x^2 - 4x)} \\ -x + 4 \\ \underline{-(-x + 2)} \\ 2 \end{array}$$

$Q(x) = x^2 + 2x - 1, \quad R(x) = 2$

TRY IT YOURSELF

1. Divide $15x^5 - 10x^3 + 5x^2$ by $5x^2$.

2. Divide $x^3 + 4x^2 + x - 6$ by $x + 2$.

3. Divide $2x^3 - x^2 + 3x + 5$ by $x - 1$.

Concept 2 | Polynomial Division

LESSON NOTES 2 • SYNTHETIC DIVISION & THEOREMS

Why Use Synthetic Division?

When the divisor is $x - a$, synthetic division is a shorter form of polynomial long division. Use the coefficients only.

$x - 3 \rightarrow a = 3$

$x + 2 = x - (-2) \rightarrow a = -2$

Synthetic Division in Four Steps

1. Write a and list the coefficients.
2. Bring down the first coefficient.
3. Multiply by a, then add.
4. Read the quotient; the last value is the remainder.

Worked Example 4 • Synthetic division

Divide $2x^3 - 3x^2 - 11x + 6$ by $x - 3$.

3	2	−3	−11	6
		6	9	−6
	2	3	−2	0

$Q(x) = 2x^2 + 3x - 2, \quad R(x) = 0$

Therefore, $2x^3 - 3x^2 - 11x + 6 = (x - 3)(2x^2 + 3x - 2)$.

Remainder Theorem When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

Worked Example 5 • Finding a remainder quickly

Find the remainder when $P(x) = x^4 - 3x^2 + 2x - 5$ is divided by $x + 2$.

$x + 2 = x - (-2)$, so $a = -2$.

$P(-2) = (-2)^4 - 3(-2)^2 + 2(-2) - 5$

$= 16 - 12 - 4 - 5 = -5$

The remainder is -5.

Factor Theorem $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Worked Example 6 • Testing a factor

Is $x - 2$ a factor of $P(x) = x^3 - 4x^2 + x + 6$?

$P(2) = 2^3 - 4(2^2) + 2 + 6 = 8 - 16 + 2 + 6 = 0$

Yes. Therefore, $x - 2$ is a factor.

$P(x) = (x - 2)(x^2 - 2x - 3)$

Worked Example 7 • Finding an unknown coefficient

When $P(x) = x^3 + kx^2 - 5x + 2$ is divided by $x - 1$, the remainder is 4. Find k.

$P(1) = 4$

$1 + k - 5 + 2 = 4$

$k - 2 = 4$, so $k = 6$

TRY IT YOURSELF

1. Use synthetic division to divide $x^3 - 2x^2 - 5x + 6$ by $x - 1$.

2. Find the remainder when $2x^3 + x^2 - 7$ is divided by $x + 2$.

3. Determine whether $x + 1$ is a factor of $x^3 + 2x^2 - x - 2$.

4. $P(x) = 3x^2 + mx - 4$ leaves remainder 6 when divided by $x - 2$. Find m.

Concept 2 | Polynomial Division

EXERCISES 1 • BASIC POLYNOMIAL DIVISION

1 Divide each polynomial by the monomial.

1. $(12x^5 - 18x^3 + 6x^2) \div 6x^2$
2. $(15a^4b^3 - 10a^2b^2 + 5ab) \div 5ab$
3. $(21m^6 - 14m^4 + 7m^2) \div 7m^2$
4. $(16y^7 + 8y^5 - 24y^3) \div 8y^3$
5. $(18p^5q^2 - 12p^3q + 6p^2q^3) \div 6p^2q$
6. $(20x^6y^2 - 15x^4y + 5x^2y^3) \div 5x^2y$

2 Rewrite in descending powers and insert zero coefficients for missing powers.

1. $x^4 - 3x + 7$
2. $5 - 2x^3 + x^5$
3. $4x^2 - 9 + x^6$
4. $7x - x^4 + 3x^3 - 2$

3 For each divisor, state the greatest possible degree of the remainder.

1. $x - 4$
2. $x^2 + 1$
3. $2x^3 - x + 1$
4. $x^4 + 5$

4 Divide and find the quotient $Q(x)$. The remainder is zero.

1. $(x^3 + 3x^2 - 4x - 12) \div (x + 3)$
2. $(2x^3 + 7x^2 + 2x - 3) \div (x + 3)$
3. $(x^4 - 5x^2 + 4) \div (x^2 - 1)$
4. $(3x^3 - x^2 - 12x + 4) \div (3x - 1)$
5. $(x^4 + x^3 - 7x^2 - x + 6) \div (x - 2)$
6. $(2x^4 - x^3 - 8x^2 + 4x) \div (x - 2)$

5 Divide and find both $Q(x)$ and $R(x)$.

1. $(x^3 - 2x + 5) \div (x - 1)$
2. $(2x^3 + 3x^2 - x + 4) \div (x + 2)$
3. $(x^4 + x^2 + 3) \div (x^2 + 1)$
4. $(3x^3 + 2x^2 - 7x + 1) \div (x - 2)$
5. $(x^4 - 2x^3 + 5x - 6) \div (x + 1)$
6. $(2x^4 + x^3 - 3x + 7) \div (x^2 + 1)$

Concept 2 | Polynomial Division

EXERCISES 2 • LONG & SYNTHETIC DIVISION

6 Use polynomial long division. Find $Q(x)$ and $R(x)$.

1. $(x^3 + 2x^2 - 7x + 4) \div (x - 2)$
2. $(2x^3 - 5x^2 + 4x - 7) \div (x - 3)$
3. $(x^4 - 3x^2 + 2x - 5) \div (x + 1)$
4. $(3x^4 - 2x^3 + x - 6) \div (x - 2)$
5. $(2x^4 + x^3 - 7x^2 + x + 3) \div (x^2 + x - 2)$
6. $(x^4 + 2x^3 - 5x^2 - 4x + 8) \div (x^2 - 2)$

7 Use synthetic division. Find $Q(x)$ and $R(x)$.

1. $(x^4 - 5x^3 + 6x^2 + 4x - 8) \div (x - 2)$
2. $(2x^4 + 3x^3 - 11x^2 - 6x + 9) \div (x + 3)$
3. $(3x^3 + x^2 - 10x - 8) \div (x + 2)$
4. $(x^5 - 1) \div (x - 1)$
5. $(2x^4 - x^3 + 5x - 6) \div (x + 1)$
6. $(4x^3 - 7x + 5) \div (x - 1)$

8 For each division, write the synthetic value a and the complete coefficient row.

1. $P(x) = 2x^4 - 3x^2 + 7$, divisor $x - 4$

   $a =$ _____ ; coefficients: _____
2. $P(x) = x^5 + 2x - 1$, divisor $x + 2$

   $a =$ _____ ; coefficients: _____
3. $P(x) = 3x^3 - x^2 + 5$, divisor $x + 4$

   $a =$ _____ ; coefficients: _____
4. $P(x) = 5 - 2x^4 + x^6$, divisor $x - 3$

   $a =$ _____ ; coefficients: _____

9 Write the best method: monomial, synthetic, or long division.

1. $(12x^4 - 6x^2) \div 3x^2$
2. $(x^4 - 5x + 2) \div (x - 3)$
3. $(2x^4 + x^2 - 1) \div (x^2 + x + 1)$
4. $(9a^5 + 3a^3) \div 3a^2$
5. $(x^5 + 1) \div (x + 1)$
6. $(3x^4 - 2x^2 + 7) \div (2x^2 - 1)$

10 Check each claimed result using $P(x) = D(x)Q(x) + R(x)$. Write Correct or Incorrect; correct any error.

1. $P = x^3 + 2x^2 - 5x - 6$; $D = x - 2$; $Q = x^2 + 4x + 3$; $R = 0$
2. $P = 2x^3 + x^2 - 4$; $D = x + 1$; $Q = 2x^2 - x + 1$; $R = -4$
3. $P = x^4 - 1$; $D = x^2 + 1$; $Q = x^2 - 1$; $R = 0$
4. $P = 3x^3 - 5x + 2$; $D = x - 1$; $Q = 3x^2 + 3x - 1$; $R = 0$

Concept 2 | Polynomial Division

EXERCISES 3 • REMAINDER & FACTOR THEOREMS

11 Use the Remainder Theorem to find the remainder. Do not divide.

1. $P(x) = x^3 - 4x + 7$; divisor $x - 2$
2. $P(x) = 2x^4 - 3x^2 + x - 5$; divisor $x + 1$
3. $P(x) = x^5 + 2x^3 - x + 4$; divisor $x - 1$
4. $P(x) = 3x^4 - 5x^2 + 2$; divisor $x + 2$
5. $P(x) = 2x^3 + x^2 - 6x + 1$; divisor $x + 3$
6. $P(x) = x^4 - 2x^3 + 5x - 8$; divisor $x - 2$

12 Use the Factor Theorem. Determine whether the stated divisor is a factor.

1. $P(x) = x^3 - 3x^2 - 4x + 12$; divisor $x - 3$
2. $P(x) = 2x^3 + x^2 - 8x - 4$; divisor $x + 2$
3. $P(x) = x^4 - 5x^2 + 4$; divisor $x - 1$
4. $P(x) = 3x^3 - 2x^2 + x - 4$; divisor $x - 1$
5. $P(x) = x^4 + x^3 - 7x^2 - x + 6$; divisor $x + 1$
6. $P(x) = 2x^4 - 3x^3 + 5x - 7$; divisor $x - 2$

13 Find the unknown coefficient using the given remainder.

1. $P(x) = x^3 + kx^2 - 2x + 5$; divisor $x - 2$; remainder 21
2. $P(x) = 2x^3 + mx - 7$; divisor $x + 1$; remainder -4
3. $P(x) = ax^2 + 3x - 2$; divisor $x - 2$; remainder 12
4. $P(x) = x^4 + bx^3 - 5$; divisor $x + 2$; remainder 3
5. $P(x) = 3x^3 - 2x^2 + cx + 1$; divisor $x - 1$; remainder 7
6. $P(x) = 2x^4 + dx^2 - 3x + 6$; divisor $x + 1$; remainder 7

14 Find the value of k so that the stated divisor is a factor.

1. $P(x) = x^3 + kx^2 - 4x + 4$; divisor $x - 2$
2. $P(x) = 2x^3 - kx^2 + 5x + 6$; divisor $x + 3$
3. $P(x) = kx^3 + 2x^2 - 5x + 1$; divisor $x - 1$
4. $P(x) = x^4 + 3x^3 + kx^2 - 4$; divisor $x + 2$
5. $P(x) = 2x^3 - 3x^2 + kx + 8$; divisor $x - 4$
6. $P(x) = kx^4 - 2x^3 + x - 4$; divisor $x + 1$

15 For each polynomial, test the listed values and write all a for which $x - a$ is a factor.

1. $P(x) = x^3 - 6x^2 + 11x - 6$; $a \in \{1, 2, 3, 4\}$
2. $P(x) = x^3 + x^2 - 4x - 4$; $a \in \{-2, -1, 1, 2\}$
3. $P(x) = 2x^3 - 3x^2 - 8x + 12$; $a \in \{-2, 1, 2, 3\}$
4. $P(x) = x^4 - 5x^2 + 4$; $a \in \{-2, -1, 1, 2\}$

Concept 2 | Polynomial Division

EXERCISES 4 • ADVANCED APPLICATIONS

16 Use the given factor, divide, then factor $P(x)$ completely.

1. $P(x) = x^3 - 6x^2 + 11x - 6$; given factor $x - 1$
2. $P(x) = x^3 + x^2 - 4x - 4$; given factor $x + 1$
3. $P(x) = 2x^3 - 3x^2 - 8x + 12$; given factor $x - 2$
4. $P(x) = x^4 - 5x^2 + 4$; given factor $x - 1$
5. $P(x) = 3x^3 - 13x^2 + 8x + 12$; given factor $x - 2$
6. $P(x) = x^4 - 10x^2 + 9$; given factor $x - 1$

17 Find the dividend $P(x)$ using $P(x) = D(x)Q(x) + R(x)$.

1. $D = x - 2$; $Q = x^2 + 3x - 1$; $R = 5$
2. $D = x + 1$; $Q = 2x^3 - x + 4$; $R = -3$
3. $D = x^2 + 1$; $Q = x^2 - 3$; $R = 2x - 1$
4. $D = 2x - 1$; $Q = x^2 + 2x + 3$; $R = 4$
5. $D = x + 3$; $Q = x^3 - 2x^2 + x - 5$; $R = 7$
6. $D = x^2 - x + 1$; $Q = 2x^2 + x - 3$; $R = x + 2$

18 Find k from the stated remainder condition.

1. $P(x) = x^2 + kx + 3$ has equal remainders for divisors $x - 2$ and $x + 1$.
2. $P(x) = 2x^3 + kx - 5$ leaves remainder 7 when divided by $x - 2$.
3. $P(x) = x^4 + kx^2 - 3$ leaves remainder 21 when divided by $x + 2$.
4. $P(x) = kx^3 - 2x + 1$ leaves remainder -5 when divided by $x + 1$.
5. $P(x) = 3x^2 + kx + 4$ leaves remainder 1 when divided by $x - 1$.
6. $P(x) = x^3 - 4x^2 + kx$ has equal remainders for divisors $x - 1$ and $x + 2$.

19 Answer using the Remainder or Factor Theorem. Do not divide.

1. If $P(3) = 0$, is $x - 3$ a factor of $P(x)$?
2. If $P(-2) = 7$, find the remainder for divisor $x + 2$.
3. If $P(1) = -4$, is $x - 1$ a factor? State the remainder.
4. If $P(0) = 5$, find the remainder for divisor x.
5. $P(4) = 0$ and the quotient for divisor $x - 4$ is $x^2 + x - 2$. Write $P(x)$ in expanded form.
6. If $P(-1) = 0$ and $P(2) = 0$, write two linear factors of $P(x)$.

20 Solve each equation using polynomial division and factorisation.

1. $x^3 - 6x^2 + 11x - 6 = 0$, knowing that $x = 1$ is a root.
2. $2x^3 - 3x^2 - 8x + 12 = 0$, knowing that $x = 2$ is a root.
3. $x^4 - 5x^2 + 4 = 0$, knowing that $x = 1$ is a root.
4. $x^3 + x^2 - 4x - 4 = 0$, knowing that $x = -1$ is a root.

Concept 2 | Polynomial Division

EXERCISES 5 • MIXED REVIEW

21 Choose the correct answer.

1. The remainder when $x^3 - 2x + 5$ is divided by $x - 1$ is:
(a) 0 (b) 2 (c) 4 (d) 6

2. For divisor $x + 3$, the synthetic value is:
(a) 3 (b) -3
(c) 1 (d) -1

3. When dividing by a quadratic polynomial, the remainder has degree at most:
(a) 0 (b) 1 (c) 2 (d) 3

4. If $P(2) = 0$, then a factor of $P(x)$ is:
(a) $x + 2$ (b) $x - 2$
(c) x (d) $2x - 1$

5. If deg $P = 5$ and deg $D = 2$, then deg Q is:
(a) 2 (b) 3
(c) 5 (d) 7

6. If $P(x) = (x + 1)Q(x) + 7$, the remainder is:
(a) -1 (b) 0 (c) 1 (d) 7

22 Complete each statement.

1. $P(x) = D(x)Q(x) +$ _____, where deg $R <$ deg _____.

2. For divisor $x + 4$, the synthetic value is $a =$ _____.

3. The complete coefficient row for $x^4 - 3x + 2$ is _____.

4. If the remainder is zero, the divisor is a _____ of $P(x)$.

5. The remainder when $P(x)$ is divided by x is $P($_____$)$.

6. If $P(-2) = 0$, then _____ is a factor of $P(x)$.

23 Write True or False. Correct each false statement.

1. Synthetic division can be used directly for divisor $x^2 + 1$.

2. The remainder from division by $x - a$ is a constant.

3. If $P(a) = 5$, then $x - a$ is a factor of $P(x)$.

4. deg Q = deg P − deg D when deg $P \geq$ deg D.

5. For divisor $x + 3$, synthetic division uses $a = 3$.

6. If $x - 1$ and $x + 2$ are factors, then $P(1) = P(-2) = 0$.

24 Identify and correct the error.

1. For divisor $x - 2$, Salma used $a = -2$ in synthetic division.

2. For $x^3 + 5x - 1$, Omar wrote the coefficient row 1, 5, -1.

3. Mona says the remainder of $P(x)$ for divisor $x - 1$ is $P(-1)$.

4. Ali says $x + 2$ is a factor because $P(2) = 0$. State the correct test.

25 Solve the mixed problems.

1. Divide $2x^4 - 3x^3 + 5x - 7$ by $x - 2$. Find $Q(x)$ and $R(x)$.

2. Find k so that $x + 1$ is a factor of $x^3 + kx^2 - 4x - 4$.

3. Find the remainder when $x^5 - 1$ is divided by $x - 1$.

4. Find $P(x)$ if $D = x^2 - 1$, $Q = 3x^2 + 2$, and $R = 4x - 5$.

5. Solve $x^3 - 4x^2 + x + 6 = 0$, knowing that $x = 2$ is a root.

6. $P(x) = x^2 + kx + 1$ has equal remainders for divisors $x - 2$ and $x + 1$. Find k.

Concept 2 | Polynomial Division

EXERCISES 6 • COMPREHENSIVE PRACTICE

26 Divide and find $Q(x)$ and $R(x)$. Use the most suitable method.

1. $(3x^5 - 5x^4 + 2x^2 - 7) \div (x - 2)$

2. $(x^5 + 2x^4 - 7x^3 - 8x^2 + 12x) \div (x + 3)$

3. $(2x^4 - 3x^2 + 5) \div (x^2 - x + 1)$

4. $(x^6 - 1) \div (x^2 + x + 1)$

5. $(4x^4 + x^3 - 9x^2 + 2x - 6) \div (2x - 1)$

6. $(x^5 - 4x^3 + 3x^2 + 7) \div (x^2 + 1)$

27 Use the Remainder or Factor Theorem.

1. Find the remainder of $2x^5 - 3x^3 + x - 4$ for divisor $x - 2$.

2. Determine whether $x + 3$ is a factor of $x^4 + 2x^3 - 9x^2 - 18x$.

3. Find k if $x - 2$ is a factor of $x^4 - kx^3 + 4x^2 - 8$.

4. Find m if $x^3 + mx^2 - 5x + 1$ leaves remainder -3 for divisor $x + 1$.

5. $P(x) = 2x^2 + kx - 3$ has equal remainders for divisors $x - 2$ and $x + 1$. Find k.

6. If $P(-1) = 0$ and $P(3) = 0$, write the quadratic factor formed by these two zeros.

28 Find the missing polynomial or value using $P(x) = D(x)Q(x) + R(x)$.

1. $D = x^2 + 2x - 1$; $Q = 2x^2 - 3x + 4$; $R = x - 5$. Find P.

2. $P = x^3 + 2x^2 - 5x + 7$; $D = x - 2$; $Q = x^2 + 4x + 3$. Find R.

3. $P = x^4 - 5x^2 + 4$; $D = x^2 - 1$; $R = 0$. Find Q.

4. $P = x^3 + x^2 - 4x - 4$; $Q = x^2 - 4$; $R = 0$. Find D.

5. $P = 2x^3 + 3x^2 + 4x + 1$; $Q = x^2 + 2x + 3$; $R = 4$. Find D.

6. $P = 3x^4 - x^2 + 4x - 7$; $D = x^2 - 1$; $R = 4x - 5$. Find Q.

29 Factor completely and solve.

1. $x^3 - 7x + 6 = 0$, knowing that $x = 1$ is a root.

2. $2x^3 + x^2 - 8x - 4 = 0$, knowing that $x = -2$ is a root.

3. $x^4 - 13x^2 + 36 = 0$.

4. $x^3 - 2x^2 - 5x + 6 = 0$, knowing that $x = 1$ is a root.

30 Solve the multi-step problems.

1. Find k, then factor $P(x) = x^3 + kx^2 - 4x - 4$ completely, given that $x + 1$ is a factor.

2. Divide $x^4 - 2x^3 + 3x - 1$ by $x + 2$; find Q and R, then verify the division identity.

3. A cubic polynomial has zeros 1, -2, and 3, with leading coefficient 2. Write it in expanded form.

4. Find a and b if $P(x) = x^3 + ax^2 + bx + 6$ has factors $x - 1$ and $x + 2$.

5. $P(x) = x^3 + ax + b$ leaves remainder 4 for divisor $x - 1$ and remainder 0 for divisor $x + 1$. Find a and b.

6. A monic cubic has factors $x - 2$ and $x + 1$, and leaves remainder 16 for divisor $x - 3$. Find $P(x)$.

Concept 3 | Applications of the Binomial Theorem

LESSON NOTES • SUBSTITUTION & IDENTITIES

Learning Objectives

- Use substitutions in the binomial expansion to prove identities.
- Evaluate weighted sums of binomial coefficients.
- Apply the Binomial Theorem to divisibility statements.

Core Strategy

Start with the expansion

$$(1 + x)^n = {}_nC_0 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_rx^r + \cdots + {}_nC_nx^n$$

Choose a suitable value of x so that the right-hand side becomes the required sum.

Useful Substitutions

Substitute	Result	Use
$x = 1$	${}_nC_0 + {}_nC_1 + \cdots + {}_nC_n = 2^n$	Sum of coefficients
$x = -1$	${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots + (-1)^n\,{}_nC_n = 0$	Alternating sum
$x = a$	$\sum_{r=0}^{n}{}_nC_ra^r = (1 + a)^n$	Weighted sum

1 Worked Example 1

Sum of the binomial coefficients

Prove that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$.

From the expansion, substitute $x = 1$:

$2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n$.

Therefore, the identity holds.

2 Worked Example 2

A weighted identity

Prove that ${}_nC_0 + 3.{}_nC_1 + 3^2.{}_nC_2 + \cdots + 3^n.{}_nC_n = 4^n$.

Substitute $x = 3$:

$(1 + 3)^n = {}_nC_0 + 3.{}_nC_1 + 3^2.{}_nC_2 + \cdots + 3^n.{}_nC_n$

Thus, the required sum equals 4^n.

3 Worked Example 3

Divisibility

Prove that $6^n - 1$ is divisible by 5 for every positive integer n.

$$6^n - 1 = (1 + 5)^n - 1$$
$$= [1 + 5.{}_nC_1 + 5^2.{}_nC_2 + \cdots + 5^n] - 1$$
$$= 5[{}_nC_1 + 5.{}_nC_2 + \cdots + 5^{n-1}]$$

Therefore, $6^n - 1$ is divisible by 5.

TRY IT YOURSELF

1. Prove that ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots + (-1)^n\,{}_nC_n = 0$.

2. Prove that ${}_nC_0 + 4 . {}_nC_1 + 4^2 . {}_nC_2 + \cdots + 4^n . {}_nC_n = 5^n$.

3. Prove that $10^n - 1$ is divisible by 9 for every positive integer n.